Function of Rules: A Case of Meaning Fluctuation

1. Introduction

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After writing down thousand and one iteration of the rule of inference (a kind of modus ponens) in conditional form by which we are supposedly allowed to follow its previous iteration, once the conditions are granted, the Tortoise said to Achilles:

> "… considering what a lot of instruction this colloquy of ours will provide for the Logicians of the Nineteenth Century…" (Lewis Carroll: What the Tortoise Said to Achilles, in: *Mind*, Vol. 416 (Oct. 1895), p. 693)

A rule for following a rule is presented as instruction for the Logicians of the Nineteenth Century but it seems that its impact goes beyond that time.

Our ability to infinitely enlist rules for following rules calls into question the function of rules. Why do we enlist them in the first place?

In *Principia Mathematica*, we can see a distinction between rule and "recognition essential to the process of deduction" which is not a rule:

> "The proofs of the earlier of the propositions of this number consist simply in noticing that they are instances of the general rules given in *1. In such cases, these rules are not premisses, since they assert any instance of themselves, not something other than their instances […] The recognition that a certain proposition is an instance of some general proposition previously proved or assumed is essential to the process of deduction from general rules, but cannot itself be erected into a general rule, since the application required is particular, and no general rule can explicitly include a particular application." (Bertrand Russell, Alfred North Whitehead: *Principia Mathematica*, vol. I., Cambridge, CUP 1910, 1927, p. 98)

Not everything we are able to say is reasonably considered to be a rule. We substitute propositions or propositional functions for variables and thus proceed to instances of rules, but the rule followed here is the rule whose instance we seek, not any rule of substitution. The same applies to derived rules, i.e. to theorems: there is a distinction between theorem and what "should be observed":

> "It should be observed that, if $p \equiv q$, q may be substituted for p without altering the truth-value of any function of p which involves no primitive ideas except those enumerated in *1. This can be proved in each separate case, but not generally, because we have no means of specifying (with our apparatus of primitive ideas) that a function is one which can be built up out of these ideas alone." (ibid. p. 115)

We observe that in propositions or propositional functions composed by negation and disjunction only, we may substitute any proposition or propositional function for something equivalent without altering the truth-value as far as this composition is concerned, but since our logic is open to any propositions and propositional functions and it is not restricted to one style of composition, we have to refrain from turning this observation into a substitution theorem.

However, the intention of Logicians kept focused on enlisting rules that supposedly allow us to follow rules. It has been embodied in textbooks, e. g. in Alonzo Church's one:

> "… This device of employing one language in order to talk about another is one for which we shall have frequent occasion not only in setting up formal languages but also in making theoretical statements as to what can be done in a formalized language, our interest in formalized languages being less often in their actual and practical use as languages than in the general theory of such use and in its possibilities in principle." (Alonzo Church: *Introduction to Mathematical Logic*, vol. I., Princeton University Press, 1956, p. 47)

In order to say what can be done with rules, we employ a distinct "language" and in this "language" we provide our instruction:

> "In setting up a formalized language we first employ as meta-language a certain portion on English. We shall not attempt to delimit precisely this portion of the English language, but describe it approximately by saying that it is just sufficient to enable us to give general directions for the manipulation of concrete physical objects (each instance or occurrence of one of the symbols of the language being such a concrete physical object, e.g. a mass of ink adhering to a bit of paper). It is thus a language which deals with matters of everyday human experience..." (ibid. p. 47 – 48)

It is by giving these "general directions for the manipulation of concrete physical objects" that we define wff and enlist the rule of substitution and the substitution theorem together with its proof. Just the physicality of our objects is actually not taken into account, we simply manipulate language expressions.

And it makes no difference if we take our objects in a set-theoretic fashion or not.

Now, a tendency towards study of rules for following rules has become so predominant and self-evident that even in historical expositions of the logic of *Principia Mathematica* with all of its distinctions, an inductive definition of propositional function and a reconstruction of substitution in lambda calculus are provided (cf. Fairouz Kamareddine, Twan Laan, Rob Nederpelt: Types in Logic and Mathematics before 1940, in: *The Bulletin of Symbolic Logic*, vol. 8/2, 2002.) and axioms of *Principia* are treated as "meta-linguistic" schemata (cf. Gregory Landini: *Russell's Hidden Substitutional Theory*. Oxford, OUP 1998.)

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2. Function of rules

Let us study the function of rules in a propositional calculus of *Principia Mathematica*-style.

As primitive (i.e. undefined) ideas, we take elementary proposition, elementary propositional function, assertion, negation and disjunction. In *Principia Mathematica*, by elementary proposition we call one not containing any apparent, i.e. bound variable.

First, we define (i.e. declare that we abbreviate) implication:

*1.01[1] $p \rightarrow q$ … is an abbreviation for … $\sim p \lor q$

As axioms, we *assert the rules* (note that we call "rules" all of our axioms):

*1.1 Anything implied by a true elementary proposition is true.

*1.2 $\vdash (p \lor p) \rightarrow p$

*1.3 $\vdash q \rightarrow (p \lor q)$

*1.4 $\vdash (p \lor q) \rightarrow (q \lor p)$

*1.6 $\vdash (q \rightarrow r) \rightarrow ((p \lor q) \rightarrow (p \lor r))$

We take the first rule, the so-called "rule of inference", as an unconditional assertion of any elementary proposition or elementary propositional function in question.

Let us see some examples of derived assertions:

*2.05 $\vdash (q \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

Proof:

*1.6 [$p \coloneqq \sim p$]	$\vdash (q \rightarrow r) \rightarrow ((\sim p \lor q) \rightarrow (\sim p \lor r))$	(1)
(1), *1.01	$\vdash (q \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$	

In this case, the derivation consists in asserting an instance of an axiom, and so, in noticing that *the rule is an instance of an already asserted rule*, and in abbreviating this propositional function according to a definition.

*2.08 $\vdash p \rightarrow p$

Proof:

*2.05 [$q \coloneqq p \lor p$, $r \coloneqq p$]		
	$\vdash ((p \lor p) \rightarrow p) \rightarrow ((p \rightarrow (p \lor p)) \rightarrow (p \rightarrow p))$	(1)
*1.2	$\vdash (p \lor p) \rightarrow p$	(2)
(1), (2), *1.1	$\vdash (p \rightarrow (p \lor p)) \rightarrow (p \rightarrow p)$	(3)
*1.3 [$q \coloneqq p$]	$\vdash p \rightarrow (p \lor p)$	(4)
(3), (4), *1.1	$\vdash p \rightarrow p$	

In this case, on the third and fifth line, we assert the succedent of an asserted implication by *following the rule* of inference. What is asserted on these lines are instances of *1.1.

[1] Číslování odkazuje k původnímu místu v PM.

Let us see a few more steps:

*2.11 $\vdash p \lor \sim p$

Proof:

*1.4 [p ≔ ~p, q ≔ p]	$\vdash (\sim p \lor p) \to (p \lor \sim p)$	(1)
*2.08, *1.01	$\vdash \sim p \lor p$	(2)
(1), (2), *1.1	$\vdash p \lor \sim p$	

*2.12 $\vdash p \to \sim\sim p$

Proof:

*2.11 [p ≔ ~p]	$\vdash \sim p \lor \sim\sim p$	(1)
(1), *1.01	$\vdash p \to \sim\sim p$	

*2.13 $\vdash p \lor \sim\sim\sim p$

Proof:

*1.6 [q ≔ ~p, r ≔ ~~~p]		
	$\vdash (\sim p \to \sim\sim\sim p) \to ((p \lor \sim p) \to (p \lor \sim\sim\sim p))$	(1)
*2.12 [p ≔ ~p]	$\vdash \sim p \to \sim\sim\sim p$	(2)
(1), (2), *1.1	$\vdash (p \lor \sim p) \to (p \lor \sim\sim\sim p)$	(3)
*2.11	$\vdash p \lor \sim p$	(4)
(3), (4), *1.1	$\vdash p \lor \sim\sim\sim p$	

In this case, we may *fail to notice that* the rules *2.11 and *2.12 *are not sufficient for the derivation*, and we also need *1.6 and the rule of inference as a means to compose them.

*2.14 $\vdash \sim\sim p \to p$

Proof:

*1.4 [q ≔ ~~~p]	$\vdash (p \lor \sim\sim\sim p) \to (\sim\sim\sim p \lor p)$	(1)
*2.13	$\vdash p \lor \sim\sim\sim p$	(2)
(1), (2), *1.1	$\vdash \sim\sim\sim p \lor p$	(3)
(3), *1.01	$\vdash \sim\sim p \to p$	

We can see that in the calculus we gradually *derive more rules from the rules* already derived or asserted as axioms. Certainly, it would be possible to assert them without a derivation or to derive them from something other than our already derived or asserted rules, but we do not intend to.

*2.2 $\vdash p \to (p \lor q)$

Proof:

*1.3 [p ≔ q, q ≔ p]	$\vdash p \to (q \lor p)$	(1)
*1.4 [p ≔ q, q ≔ p]	$\vdash (q \lor p) \to (p \lor q)$	(2)

$*2.05\ [q \coloneqq q \lor p, r \coloneqq p \lor q]$

$\vdash ((q \lor p) \to (p \lor q)) \to ((p \to (q \lor p)) \to (p \to (p \lor q)))$ (3)

(2), (3), *1.1 $\vdash (p \to (q \lor p)) \to (p \to (p \lor q))$ (4)

(1), (4), *1.1 $\vdash p \to (p \lor q)$

Such a *derivation by following a rule we abbreviate*

$*1.3\ [p \coloneqq q, q \coloneqq p]$ $\vdash p \to (q \lor p)$ (1)

$*1.4\ [p \coloneqq q, q \coloneqq p]$ $\vdash (q \lor p) \to (p \lor q)$ (2)

(1), (2), *2.05 $\vdash p \to (p \lor q)$

where the last line means: by following the rule of inference in two successive steps, we assert the succedent of a suitable instance of the rule *2.05. We follow this rule, as if we have derived the unconditional rule “Anything is implied by anything implying something implying the first thing.”

... Etc.

... opačný postup: hledání axiomů?

Hence, the function of rules includes: we *assert them*; we notice that *they are instances of already asserted rules*; we assert something *by following them*; we notice that *some of them are not sufficient* for a derivation; we *derive more rules from them*; we *abbreviate a derivation following them*. There is no reason to consider some parts of this function more important than others.

The function of rules is given by our intention: we intend to shew that we can derive many laws of logic from a small number of primitive ideas and axioms.

3. Missing rule

Let us study what happens with the function of rules when we try to add the rule of substitution:

We can see this in one of our examples:

*2.14 $\vdash \sim\sim p \to p$

Proof:

$*1.4\ [q \coloneqq \sim\sim\sim p]$ $\vdash (p \lor \sim\sim\sim p) \to (\sim\sim\sim p \lor p)$ (1)

*2.13 $\vdash p \lor \sim\sim\sim p$ (2)

(1), (2), *1.1 $\vdash \sim\sim\sim p \lor p$ (3)

(3), *1.01 $\vdash \sim\sim p \to p$

We observe that on the third line we have followed the rule of inference and we tend to interpret this rule as a statement *on which conditions we are allowed to do so*:

*1.1a[2] If we already have „⊦ φp“ and „⊦ φp → ψp“, we may proceed to „⊦ ψp“.

Then, the procedure of proof looks (with the assertions of propositional functions remaining the same):

Proof:

*1.4 [q ≔ ~~~p] (1)

*2.13 (2)

(1), (2), *1.1a (3)

(3), *1.01

But now, we will miss rules stating on which conditions we are allowed to make other steps. Either we take our assertions as schemata, or we add the rule of substitution:

*1.1b If we already have „⊦ φp“ and if ψp is an instance of φp, we may proceed to „⊦ ψp“.

Here, for the sake of simplicity, we consider also abbreviated or expanded forms of φp, and we take an elementary propositional function itself as its own instance. Then, the procedure of proof looks:

Proof:

*1.4, *1.1b (1)

*2.13, *1.1b (2)

(1), (2), *1.1a (3)

(3), *1.01, *1.1b

However, the rule of substitution states two conditions, and we can enlist the second one as well:

P1 (p ∨ ~~~p) → (~~~p ∨ p) … is an instance of … (p ∨ q) → (q ∨ p).

P2 p ∨ ~~~p … is an instance of … p ∨ ~~~p.

P3 ~~p → p … is an instance of … ~~~p ∨ p.

Now, we derive consequences *from these assertions about expressions*:

Proof:

*1.4, P1, *1.1b (1)

*2.13, P2, *1.1b (2)

(1), (2), *1.1a (3)

(3), P3, *1.1b

And we can state the relation of “being an instance of” as an assertion *about manipulation of language expressions*:

*1.02a ψp is an instance of φp … is an abbreviation for … ψp results by substitution of any elementary propositional function for each occurrence of some variable in φp.

[2] Písmeny v číslování označuji alternativní a přidaná tvrzení.

(or again, for the sake of simplicity, in an abbreviated or expanded form of φp; and we may consider similar definition using simultaneous substitution for several variables).

Indeed, we can state that "we already have" something, i.e. the first condition of the rule of substitution and both conditions of the rule of inference (in the sense of *1.1a), as an assertion about language expressions:

*1.02b We already have "⊦ φp" … is an abbreviation for … φp is in the list of axioms and already derived elementary propositional functions.

And finally, we explicitly enlist the conditions

P4 We already have "⊦ (p ∨ q) → (q ∨ p)"

P5 We already have "⊦ p ∨ ∼∼∼p"

in the proof

Proof:

P4, P1, *1.1b (1)

P5, P2, *1.1b (2)

(1), (2), *1.1a (3)

(3), P3, *1.1b

This procedure of proof is composed of assertions about language expressions only, and its own rules (by following which we possibly assert instances of *1.1b, *1.1a or of their succedents … etc.) are tacit.

Hence, the function of rules includes: we state by them, *on which conditions we are allowed to do something*, and we explicitly enlist them as assertions *about language expressions* or *about manipulation of language expressions*.

The function of rules fluctuated, as our intention fluctuated: we do not only intend to shew that we can derive something from something, but we also intend to say, how we are allowed to do this by manipulating language expressions.

4. Missing theorem

Let us study what more happens with the function of rules when we try to derive the substitution theorem.

Again, we can see this in one of our examples:

*2.2 ⊦ p → (p ∨ q)

Proof:

*1.3 [p ≔ q, q ≔ p] ⊦ p → (q ∨ p) (1)

*1.4 [p ≔ q, q ≔ p]	$\vdash (q \vee p) \rightarrow (p \vee q)$	(2)
(1), (2), *2.05	$\vdash p \rightarrow (p \vee q)$	

When looking for this derivation, we may fail to notice that we need some rules besides *1.3 and *1.4 to compose them. This is because of a tacit suggestion of the substitution theorem:

*4.03a If we already have "$\vdash p \leftrightarrow q$", we may proceed to "$\vdash \varphi p \leftrightarrow \varphi q$".

or better, of its consequence

*4.03b If we already have "$\vdash p \leftrightarrow q$" and "$\vdash \varphi p$", we may proceed to "$\vdash \varphi q$".

(We presuppose here that we already have definitions

*3.01 $p \wedge q$ … is an abbreviation for … $\sim(\sim p \vee \sim q)$

*4.01 $p \leftrightarrow q$ … is an abbreviation for … $(p \rightarrow q) \wedge (q \rightarrow p)$

and a derived assertion

*3.26 $\vdash (p \wedge q) \rightarrow p$

The derivation of *4.03b from *4.03a then looks

cond.	"$\vdash p \leftrightarrow q$"	(1)
cond.	"$\vdash \varphi p$"	(2)
(1), *4.03a	"$\vdash \varphi p \leftrightarrow \varphi q$"	(3)
(3), *4.01	"$\vdash(\varphi p \rightarrow \varphi q) \wedge (\varphi q \rightarrow \varphi p)$ "	(4)
(4), *3.26	"$\vdash \varphi p \rightarrow \varphi q$"	(5)
(2), (5), *1.1	"$\vdash \varphi q$")	

The suggested proof (with the tacit theorem enlisted) would look like:

*1.3 [p ≔ q, q ≔ p]	$\vdash p \rightarrow (q \vee p)$	(1)
*4.31	$\vdash (q \vee p) \leftrightarrow (p \vee q)$	(2)
(1), (2), *4.03b	$\vdash p \rightarrow (p \vee q)$	

(where *4.31 is an already derived assertion), and it more or less articulates the idea "the asserted proposition is just *1.3 with the disjunction in succedent permuted, and we may do this".

We observe that the substitution theorem shortcuts a specific procedure of proof. And because we can provide similar shortcuts for similar cases, we try to put forth more general instruction (by the induction on the length of φp) for a transition from "$\vdash p \leftrightarrow q$" to "$\vdash \varphi p \leftrightarrow \varphi q$":

a) In case φp is an *atomic* elementary proposition, it is either atomic elementary proposition p or some other atomic elementary proposition r. In the former case we simply assert

cond.	"$\vdash p \leftrightarrow q$"	(1)
(1)	"$\vdash p \leftrightarrow q$", which is "$\vdash \varphi p \leftrightarrow \varphi q$"	

and in the latter case

cond.	"$\vdash p \leftrightarrow q$"	(1)
*4.2	"$\vdash r \leftrightarrow r$", which is "$\vdash \varphi p \leftrightarrow \varphi q$"	

b) In case φp is a compound elementary proposition and it is composed by negation of an elementary proposition ψp for which we already have instruction, we assert

cond.	"$\vdash p \leftrightarrow q$"	(1)
(1), cond.	"$\vdash \psi p \leftrightarrow \psi q$"	(2)
(2), *4.11	"$\vdash \sim\psi p \leftrightarrow \sim\psi q$", which is "$\vdash \varphi p \leftrightarrow \varphi q$"	

c) In case φp is a compound elementary proposition and it is composed by disjunction of elementary propositions $\psi_1 p$ and $\psi_2 p$ for which we already have instruction, we assert

cond.	"$\vdash p \leftrightarrow q$"	(1)
(1), cond.	"$\vdash \psi_1 p \leftrightarrow \psi_1 q$"	(2)
(1), cond.	"$\vdash \psi_2 p \leftrightarrow \psi_2 q$"	(3)
(2), (3), *3.2	"$\vdash (\psi_1 p \leftrightarrow \psi_1 q) \wedge (\psi_2 p \leftrightarrow \psi_2 q)$"	(4)
(4), *4.39	"$\vdash (\psi_1 p \vee \psi_2 p) \leftrightarrow (\psi_1 q \vee \psi_2 q)$", which is "$\vdash \varphi p \leftrightarrow \varphi q$"	

(We presuppose here that we already have the assertions

*3.2 $\vdash p \rightarrow (q \rightarrow (p \wedge q))$

*4.11 $\vdash (p \leftrightarrow q) \leftrightarrow (\sim p \leftrightarrow \sim q)$

*4.2 $\vdash p \leftrightarrow p$

*4.39 $\vdash ((p \leftrightarrow q) \wedge (r \leftrightarrow s)) \rightarrow ((p \vee r) \leftrightarrow (q \vee s))$

derived.)

In the suggested proof of *2.2, our instruction for the step following the substitution theorem would look like

*4.31	$\vdash (p \vee q) \leftrightarrow (q \vee p)$	(1)
*4.2	$\vdash \sim p \leftrightarrow \sim p$	(2)
(1), (2), *3.2	$\vdash (\sim p \leftrightarrow \sim p) \wedge ((p \vee q) \leftrightarrow (q \vee p))$	(3)
(3), *4.39	$\vdash (\sim p \vee (p \vee q)) \leftrightarrow (\sim p \vee (q \vee p))$	(4)
(4), *1.01	$\vdash (p \rightarrow (p \vee q)) \leftrightarrow (p \rightarrow (q \vee p))$	

In order to explicate this inductive instruction, we need to *talk about atomicity and composition of language expressions*. Our primitive ideas must pertain to this.

However, even then we cannot proceed this way to any elementary propositional function φp. Apart from elementary propositional functions of propositions, whose truth-value does not depend only upon the truth-values of their arguments (i. e. so called intensional functions), we cannot proceed to elementary propositional functions such as p & q, which would be equivalent to conjunction with respect to its truth-value, but would not be an abbreviation for $\sim(\sim p \vee \sim q)$.

Of course, we may assert (without proof)

$\vdash (p \,\&\, q) \leftrightarrow (p \wedge q)$

according to our instruction derive

$\vdash (p \leftrightarrow q) \rightarrow ((r \wedge p) \leftrightarrow (r \wedge q))$

and then

$\vdash (p \leftrightarrow q) \rightarrow ((r \,\&\, p) \leftrightarrow (r \,\&\, q))$

(and by analogy, for the first term), but according to our instruction only, we cannot proceed from $p \leftrightarrow q$ to $(r \,\&\, p) \leftrightarrow (r \,\&\, q)$.

But we tend to say that we can proceed to any elementary propositional function we employ, i. e. add a rule

*4.03c We do not employ elementary propositions that are not composed by negation and disjunction of atomic elementary propositions only.

by which we *restrict our intention* to specific language expressions. Then, our instruction appears to be a derivation of the substitution theorem.

Hence, the function of rules includes: we state by them, *which language expressions are atomic and which are compound*, and we use them to *restrict our intention* this way.

The function of rules fluctuated further, as our intention fluctuated further: we do not only intend to shew that we can derive something from something and say how we are allowed to do this by manipulating language expressions, but we intend to restrict our intention this way.

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5. Závěr

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Posun v úloze pravidel je skrytý, přestože posun našeho zájmu v logice přiznáváme.

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